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FUNDAMENTAL FREQUENCIES OF MOON- AND LENS-SHAPED MEMBRANES

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Vibration of membranes is important in the generation and reception of sound [1]. The governing Helmholtz equation also describes simply-supported vibrating plates and electromagnetic wave guides [2]. The solution methods and the variety of boundary shapes studied were reviewed by Ng [3]. The purpose of this note is to present the results for the moon-shaped and lens-shaped membranes which have never been investigated before.

Let the boundary of the membrane be described by two circular arcs. Let the left arc be described by

$$x^2 + y^2 = 1, \qquad -1 \leqslant x \leqslant c < 1, \tag{1}$$

where all lengths have been normalized by the radius. The right arc is given by

$$b(x^2 + y^2) + [1 - (c + b)^2]x + (c + b)[c(c + b) - 1] = 0, \tag{2}$$

where c is the horizontal co-ordinate of the intersecting points $(c, \pm\sqrt{1 - c^2})$ and $(b + c, 0)$ is the midpoint of the right arc. When $b = 0$ the right arc is a vertical line segment at $x = c$. Figure 1 shows the membrane is lens-shaped when $b > 0$ and moon-shaped when $b < 0$.

The governing equation is

$$w_{xx} + w_{yy} + k^2 w = 0, \tag{3}$$

where $w(x, y)$ is the normalized displacement, and k is the frequency normalized by (tension per length/density)$^{1/2}$/length. On the boundary, $w = 0$. The fundamental frequency is the lowest eigenvalue k.

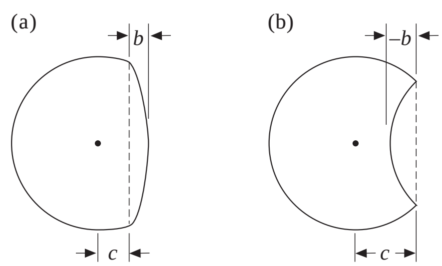

Figure 1. The geometry: (a) lens-shaped, (b) moon-shaped membranes.

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Due to the complex boundary, there is no closed-form solution and finite differences or elements are very tedious. The variational method described by Weinstock [4] will be used. The solution of equation (3) minimizes the integral

$$I = \iint (w_x^2 + w_y^2)\,\mathrm{d}\sigma, \tag{4}$$

with the constraint

$$\iint w^2\,\mathrm{d}\sigma = 1, \tag{5}$$

where σ is the area bounded by the membrane boundary. One approximates w by the expansion

$$\begin{aligned} w(x, y) = {} & (x^2 + y^2 - 1)\{b(x^2 + y^2) + [1 - (c + b)^2]x + (c + b)[c(c + b) - 1]\} \\ & \times (a_1 + a_2 x + a_3 x^2 + a_4 y^2 + a_5 x^3 + a_6 x y^2 + a_7 x^4 + a_8 x^2 y^2 + a_9 y^4 + \cdots) \\ \equiv {} & \sum_1^N a_i \phi_i(x, y). \end{aligned} \tag{6}$$

Here w satisfies the zero boundary conditions and the series is even in y, complete, and converge within the circle $x^2 + y^2 = 1$. N can be taken as 1, 2, 4, 6, 9 etc. The eigenvalue k is obtained from

$$|\Gamma_{ij} - k^2 \Lambda_{ij}| = 0, \tag{7}$$

where

$$\Gamma_{ij} = \iint (\phi_{ix}\phi_{jx} + \phi_{iy}\phi_{jy})\,\mathrm{d}\sigma, \qquad \Lambda_{ij} = \iint \phi_i \phi_j\,\mathrm{d}\sigma. \tag{8, 9}$$

First one tests the accuracy of the present method using the results for the semicircular membrane, obtained by separation of variables [1]. The exact eigenvalue is the first zero of the Bessel function J_1, i.e., 3·8317. The present numerical values for this case, using different numbers of terms, are given in Table 1.

One sees that $N = 4$ is sufficient to guarantee an error less than 0·1%, and sometimes $N = 6$ has been used to assure convergence.

The range of parameters used is $-1 < c < 1$ and $-1 - c < b < \min(1 - c, \sqrt{1 - c^2})$. Note that $c < 0$, $b \geqslant \sqrt{1 - c^2}$ cases are redundant since one can use the mirror image of the $c > 0$ cases.

Table 2 lists the fundamental frequency for the circular segment membrane ($b = 0$).

The results for the moon-shaped ($b > 0$) and lens-shaped ($b < 0$) membranes are plotted in Figure 2.

TABLE 1

Convergence of variational approach

N	1	2	4	6	exact
k	4·000	3·883	3·833	3·832	3·8317

TABLE 2

Fundamental frequency for the circular segment membrane ($b = 0$)

c	-1	$-0·8$	$-0·6$	$-0·4$	$-0·2$	0	0·2	0·4	0·6	0·8	1
k	∞	17·028	8·780	6·026	4·651	3·832	3·295	2·924	2·665	2·497	2·405

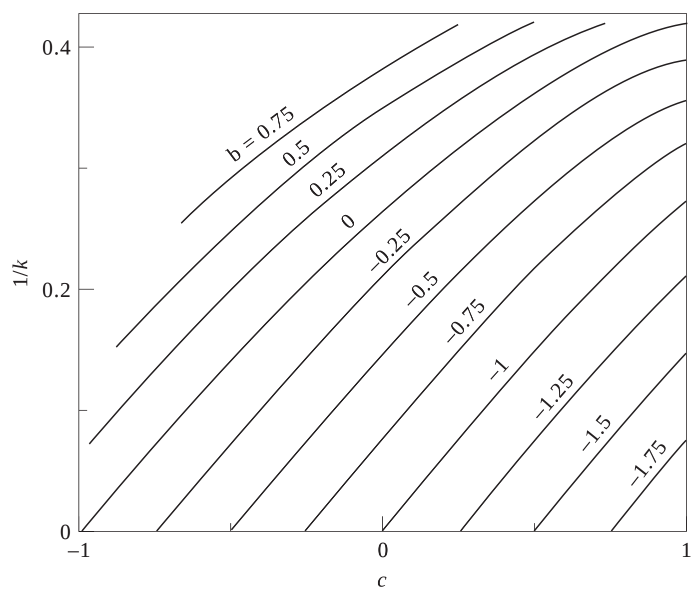

Figure 2. Normalized period $1/k$ as a function of b and c.

The normalized period $(1/k)$ is plotted, since the normalized frequency k would span too large a range. The $b > 0$ curves begins at $\sqrt{1-c^2}$ and ends at $1 - c$, where the period is 1/2·405.

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